SECTION 6.7: PHYSICAL APPLICATIONS

WORK: Basic principles:

- Work = (force)(distance)
- Work is additive.
- Measurements: Force: lbs or N; Work: ft-lbs, J
- Hooke's Law: the force F required to stretch or compress a spring is directly proportional to the displacement of the spring x: $F = kx$, where k is a constant of proportionality.
- Weight is a force: $w = mg$; where g is the acceleration due to gravity: $g = 32ft/s^2$ or $g = 9.8m/s^2$
- constant force; solid object: multiply (do example with 5 lb book lifted 3 feet; 0.5 N of water lifted 1 meter.

EXAMPLE 1: A force of 10 pounds stretches a spring 5 feet.

1. Find the spring constant, k.

   Ans: $k = 2$ pounds per foot.

2. Find the work done stretching the spring 5 feet from its equilibrium position.

   Ans: $W = \int_0^5 2x\,dx = 25$ ft-lbs

3. Find the work done stretching the spring an additional 20 feet.

   Ans: $W = \int_5^{25} 2x\,dx = 600$ ft-lbs

EXAMPLE 2: The force of gravity, F, between two objects of masses M and m a distance r apart is given by:

$$F = G\frac{Mm}{r^2},$$

where $G = 6.67 \times 10^{-11}\ \dfrac{\text{N m}^2}{\text{kg}^2}$ is the gravitational constant.

Assume Earth is a sphere of radius 6371 km with a mass of 5.972×10^{24} kg, all of which is concentrated at its center. Find the work done against gravity launching a 500 kg satellite into an orbit of 1000 km above the Earth.

Ans: Converting distances to *meters*: $W = \displaystyle\int_{6371000}^{7371000} (6.67 \times 10^{-11})\frac{(5.972 \times 10^{24})(500)}{r^2}\,dr = 4.241 \times 10^9$ Joules

EXAMPLE 3: A cylindrical septic tank measures 10 feet in diameter across the top of the tank and is 10 feet deep. If the tank is buried so that the top if the tank is 3 feet below ground level, find the work done emptying the tank by pumping the sewage out from the top. (Assume sewage weighs 50 pounds per cubic foot.)

Ans: Assuming $y = 0$ is the bottom of the tank: $W = \displaystyle\int_0^{10} 50(25\pi)(13 - y)\, dy = 100000\pi \approx 314159$ ft-lbs

EXAMPLE 4: (VIDEO) A tank in the shape of an inverted cone has a height of 20 m and a base radius of 5 m and is filled with water to a depth of 15 m. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is 1000kg/m^3. (HINT: Remember Isaac!)

Ans: Assuming $y = 0$ is the bottom of the tank: $W = \displaystyle\int_0^{15} (20 - y)(9800)\left[\pi\left(\frac{y}{4}\right)^2\right] dy \approx 1.89 \times 10^7$ Joules

EXAMPLE 5: (VIDEO) A 10 ft. chain weighing 2 lbs. / ft. is suspended vertically from a winch 15 ft. from the ground.

1. Find the work done reeling in 7 feet of the chain.

   Ans: Assuming $y = 0$ is the top of the chain:

   $$W = \int_0^7 2y\, dy + (3\,\text{feet of chain})(2\,\text{pounds per foot})(7\,\text{feet moved}) = 91\,\text{ft-lbs}$$

2. Find the work done reeling in 7 feet of the chain with an 850 pound Snorlax tied to the end.

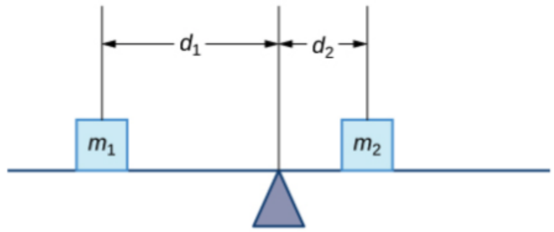
   Ans: Add (850 pounds)(7 feet) = 5950 ft-lbs to part (a) to get: 6041 ft-lbs.

HOMEWORK: Section 6.7: 23, 27, 31, 33, 35, 37, 39, 69

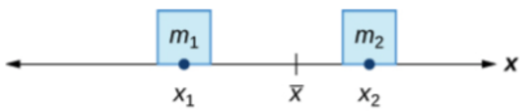
MOMENTS AND CENTER OF MASS (AS TIME PERMITS)

ONE DIMENSIONAL SYSTEMS: DISCRETE SYSTEMS

BIG IDEA: Suppose we put two masses m_1 and m_2 on a board as pictured below and wish to place a fulcrum between the masses so as to balance the board.



For the board above to balance at the fulcrum, we need that $m_1 d_1 = m_2 d_2$. This is called the **principle of the lever**. To help us find where to place the fulcrum, we model the scenario more precisely. We imagine the board as the x-axis with a mass of m_1 located at the point x_1 and a mass of m_2 located at the point x_2. Our goal is to find $\overline{x}$, the so-called **center of mass** of the system. In many situations, such as straight-line kinematics (momentum and kinetic energy), problems involving a system of masses can be reduced to a simpler problem involving the total mass of the system located at the center of mass.



Using the principle of the lever, we know that:

$$
\begin{aligned}
m_1 d_1 &= m_2 d_2 \\
m_1|\overline{x} - x_1| &= m_2|\overline{x} - x_2| \\
m_1(\overline{x} - x_1) &= m_2(x_2 - \overline{x}) \\
m_1\overline{x} - m_1 x_1 &= m_2 x_2 - m_2\overline{x} \\
m_1\overline{x} + m_2\overline{x} &= m_1 x_1 + m_2 x_2 \\
\overline{x}(m_1 + m_2) &= m_1 x_1 + m_2 x_2 \\
\overline{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}
\end{aligned}
$$

The quantity $m = m_1 + m_2$ is the **total mass** of the system. The quantity $M = m_1 x_1 + m_2 x_2$ is called the **moment of the system about the origin**. In general, a moment is a mass times a displacement. So in this case, $m_1 x_1$ is the moment of m_1 about the origin since we are multiplying m_1 about its displacement from $x = 0$.

In general, if we have n masses $m_1, m_2, \ldots m_n$ located at $x_1, x_2, \ldots x_n$, then:

$$
m = \sum_{k=1}^{n} m_k, \quad M = \sum_{k=1}^{n} m_k x_k, \quad \overline{x} = \frac{M}{m}
$$

EXAMPLE 1: Consider the system of masses:

$m_1 = 10$ at $x_1 = -3$, $m_2 = 5$ at $x_2 = 1$, and $m_3 = 2$ at $x_3 = 4$. Find the center of mass, $\overline{x}$.

Ans: $m = 17$, $M = -17$ so $\overline{x} = -1$.

ONE DIMENSIONAL SYSTEMS: CONTINUOUS SYSTEMS

Suppose we model a wire as an interval $[a, b]$ and let $\rho(x)$ be the **linear density** of the wire. This means that $\rho(x)$ tells you the density of the wire, with units of mass per length, at location x.

Using the old 'chop and add' approach to integration, we can show that:

$$m = \int_a^b \rho(x)\,dx, \quad M = \int_a^b x\,\rho(x)\,dx, \quad \overline{x} = \frac{M}{m}$$

EXAMPLE 2: A wire is modeled by $[0, 1]$ with density $\rho(x) = e^{-x}$. Find the center of mass of the wire.

Ans: $m = 1 - e^{-1} = \dfrac{e-1}{e}$, $M = 1 - 2e^{-1} = \dfrac{e-2}{e}$, so $\overline{x} = \dfrac{e-2}{e-1}$.

TWO DIMENSIONAL SYSTEMS: DISCRETE SYSTEMS

Suppose we locate a mass m_1 at the point (x_1, y_1). Then we define:

- M_x is the **moment about the** x**-axis**.

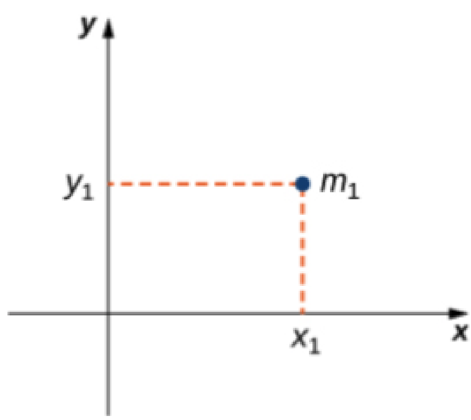
  The moment about the x-axis is the product of the mass m_1 with its displacement from the x-axis.

  Hence, $M_x = m_1 y_1$.

- M_y is the **moment about the** y**-axis**.

  The moment about the y-axis is the product of the mass m_1 with its displacement from the y-axis.

  Hence, $M_y = m_1 x_1$.



In general, if we have n masses $m_1, m_2, \ldots m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, then:

$$m = \sum_{k=1}^{n} m_k, \quad M_x = \sum_{k=1}^{n} m_k y_k, \quad M_y = \sum_{k=1}^{n} m_k x_k, \quad (\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

EXAMPLE 3: Consider the system of masses:

$m_1 = 10$ at $(x_1, y_1) = (-3, 2)$, $m_2 = 5$ at $(x_2, y_2) = (1, 3)$, and $m_3 = 2$ at $(x_3, y_3) = (4, -5)$.

Find the center of mass, $(\overline{x}, \overline{y})$.

Ans: $m = 17$, $M_x = 25$, $M_y = -17$ so $(\overline{x}, \overline{y}) = \left(-1, \dfrac{25}{17}\right)$

TWO DIMENSIONAL SYSTEMS: CONTINUOUS SYSTEMS

Suppose we have a wire modeled by the graph of a rectifiable curve $y = f(x)$ over the interval $[a, b]$ with linear density $\rho(x)$. Using the old 'chop and add' approach approach, we can show that:

$$m = \int_a^b \rho(x)\sqrt{1+[f'(x)]^2}\,dx, \quad M_y = \int_a^b x\,\rho(x)\sqrt{1+[f'(x)]^2}\,dx, \quad M_x = \int_a^b f(x)\,\rho(x)\sqrt{1+[f'(x)]^2}\,dx$$

so that the center of mass is $(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$.

EXAMPLE 4: Suppose a wire is modeled by the graph of $y = f(x) = x^2$ over $[0, 1]$.

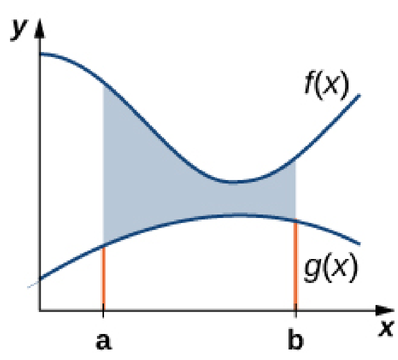
Assuming a constant density $\rho(x) = k$, set-up integrals which would calculate m, M_y and M_x and use a graphing utility to evaluate these integrals to find the center of mass, rounded to four decimal places.

Ans: $m = \displaystyle\int_0^1 k\sqrt{1+4x^2}\,dx = 1.4789...$, $M_y = \displaystyle\int_0^1 kx\sqrt{1+4x^2}\,dx = .8483...$, $M_x = \displaystyle\int_0^1 kx^2\sqrt{1+4x^2}\,dx = .6063...$

Hence, $(\overline{x}, \overline{y}) \approx (.5736, .4100)$

TWO DIMENSIONAL SYSTEMS: PLANAR LAMINA

Suppose we model **planar lamina** as a region R in the xy-plane bounded by the graphs of two continuous functions $y = f(x)$ and $y = g(x)$ as depicted below.



$\rho(x)$ be the **planar density** of the region. This means that $\rho(x)$ tells you the density of the lamina, with units of mass per area, at location x.

Using the old 'chop and add' approach approach, we can show that:

$$m = \int_a^b \rho(x)\,[f(x)-g(x)]\,dx, \quad M_y = \int_a^b x\,\rho(x)\,[f(x)-g(x)]\,dx, \quad M_x = \int_a^b \rho(x)\left[\frac{[f(x)]^2-[g(x)]^2}{2}\right]dx$$

EXAMPLE 5: Assuming a constant density $\rho(x) = 1$, find the center of mass of a semicircle modeled as the region bounded by $y = f(x) = \sqrt{r^2 - x^2}$ and $y = g(x) = 0$.

Ans: $m = \displaystyle\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{2}$, $M_y = \displaystyle\int_{-r}^{r} x\sqrt{r^2 - x^2}\, dx = 0$, $M_y = \displaystyle\int_{-r}^{r} \frac{(\sqrt{r^2 - x^2})^2}{2}\, dx = \frac{2r^3}{3}$

Hence, $(\overline{x}, \overline{y}) = \left(0, \dfrac{4r}{3\pi}\right)$.

HOMEWORK: 13 - 19 odd